DYNAMICS OF OPEN-CHAINED SYSTEMS OF RIGID BODIES

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ABSTRACT

The equations of motion for multi-body open-chained mechanical systems containing multiple degrees-of-freedom revolute joints and rigid bodies are formulated in a unique manner based on a form of Lagrange's equations for quasi-coordinates (Boltzmann/Hamel equations). The equations are shown to be more readily formulated in terms of inertial coordinates rather than "relative" (to an adjacent body) coordinates and the resulting equations are shown to be more computationally efficient. A method of transforming the equations into a form in terms of joint coordinates is presented so that joint constraints may be enforced and the equations of motion may be reduced to minimum order.

1. Introduction

In this paper the dynamic modeling of multibody systems consisting of open chains of rigid bodies is addressed. Adjacent bodies are connected by means of a revolute joint which permits relative rotations of one, two or three degrees of freedom. The motivation for this work is the dynamic modeling of robots and spacecraft, such as the planned Space Station Freedom, for the purposes of simulation and control.

Much work has been done on the problem of modeling multibody systems. In the mid 1960's a formalism was being developed for formulating the equations of motion of systems of interconnected rigid bodies arranged in a "topological tree" (Hooker and Margulies, 1965 and Roberson and Wittenburg, 1967). Software packages were developed for the analysis of multibody problems: NBOD by Frisch (1974) for rigid-body systems and DISCOS by Bodley et al. (1978) for structurally flexible systems. Other commercial packages have been developed more recently. Also more recently, Huston et al. (1979, 1980, 1981) presented a series of papers regarding the rigid and flexible body effects in multibody system dynamics. Sunada and Dubowsky (1983) and Shabana and Wehage (1984) have also treated the case of multibody systems with structural flexibility.

There are two problems in dynamics that may need to be solved, the control problem and the simulation problem. In the control problem a trajectory is given and the required joint torques and forces are to be determined. Whereas in the simulation problem, given torque and force histories, the equations of motion are integrated to determine the motion of the system in time.

To formulate the control and simulation problems, the dynamic equations of motion of the system must be developed. There are many techniques available for formulating the equations of motion for multi-body systems. Paul (1975) reviewed five of the most commonly used methods to describe the dynamics of two and three dimensional mechanisms. Some of the most common methods include formulations based on the Newton-Euler equations, d'Alembert's Principle and Lagrange's equations.

In some methods the equations governing the motion of each body are coupled by means of Lagrange multipliers. The use of Lagrange multipliers to enforce kinematic constraints increases the order of the system. Likewise, in the Newton-Euler method, the inclusion of constraint forces and moments at the joints increases the order of the system. If Lagrange's equations are applied in terms of true joint coordinates, the resulting equations are of minimal order, however, the math is tedious and simplifications may be difficult to discover. With d'Alembert's Principle, the resulting equations also may be of minimum order, however, analytical expressions for accelerations of the bodies must be developed, whereas with Lagrange's equations, this is not the case.

While the validity of the equations of motion of a dynamic system does not depend on the particular method of formulation, the form of the resulting equations may be different. The best method of equation of motion development for a particular dynamic system produces the simplest and most useful (e.g. symmetry preservation and form invariance) form of the equations in an orderly fashion.

In this paper the equations of motion of open-chained, rigid-body, multibody systems are formulated using a form of Lagrange's equations for quasi-coordinates also known as the Boltzmann/Hamel equations (Neimark and Fufaev, 1972 and Meirovitch, 1970). The quasi-coordinates are defined such that their time derivatives are the angular velocities of the bodies. A new form of these equations was introduced by Quinn (1990) based on two new identities which eased their implementation. This form of the Boltzmann/Hamel equations permits a straightforward and orderly formulation of the equations of motion.

Ju and Mansour (1989) demonstrated with examples that, for a simple three degree of freedom PUMA robot, the equations of motion in terms of inertial coordinates (measured from an inertial reference frame) are computationally superior to those in terms of relative coordinates (measured relative to an adjacent reference frame). We show analytically that this is indeed the case and, in fact, that inertial coordinates permit a more straightforward formulation of the equations of motion for general three dimensional multibody dynamic systems.

In general when inertial coordinates are used, there are three coordinates describing the orientation of each body (e.g. Euler angles or quasi-coordinates). When a system contains joints with less than three degrees of freedom, the formulation remains useful for the control problem. However, for the simulation problem, kinematic joint constraints must be imposed. As an alternative to the Lagrange multiplier approach, the equations may be transformed into configuration space and the transformed equations reduced in order, according to the constraints. Hence, we may develop equations of motion of minimum order, with symmetry preserved, by means of a transformation of the quasi-coordinate formulation.

The control and simulation methods are demonstrated by means of numerical examples for two cases, a planar robot and a spatial manipulator with six degrees of freedom.

2. Generalized Equations of Motion in Terms of Quasi-Coordinates

"Generalized" refers to an open-chained system consisting of n rigid bodies connected by revolute joints, each with three degrees-of-freedom. In robotic systems, these n bodies represent n links shown along with attached coordinate systems in Fig. 1. The body frame i is fixed to link i at joint i. The position vector from joint i to joint i+1 is defined as $\underset{\sim}{L}_i$. Each body i, in its rigid state, has three rotational degrees of freedom.

Consider the structure to be undergoing large joint motion and define $\underset{\sim}{r}_i$ as the position vector of a particle P on body i with respect to the inertial frame which can be expressed as

$$\underset{\sim}{r}_i = \underset{\sim}{R} + \sum_{j=1}^{i-1} C_{Nj} \, \underset{\sim}{L}_j + C_{Ni} \, \underset{\sim}{\rho}_i \tag{1}$$

where $\underset{\sim}{R}$ is the position vector of the base and $\underset{\sim}{\rho}_i$ is the position vector of particle P relative to joint i. C_{Ni} is the transformation matrix describing the inertial orientation of the i-th body frame and is a function of parameters representing rotations such as Euler angles, Euler parameters, or joint coordinates. The inertial angular velocity of link i $\underset{\sim}{\omega}_i$ is the time derivative of the inertial quasi-coordinates $\underset{\sim}{\beta}_i$ or $\underset{\sim}{\omega}_i = \dot{\underset{\sim}{\beta}}_i$.

The velocity of point P is the time derivative of Eq. (1), which can be expressed in matrix form as

$$\dot{\underset{\sim}{r}}_i = \dot{\underset{\sim}{R}} + \sum_{j=1}^{i-1} (\dot{C}_{Nj}\underset{\sim}{L}_j) + \dot{C}_{Ni}\underset{\sim}{\rho}_i \tag{2}$$

Note that Eq. (2) contains time derivatives of transformation matrices which can be expressed as

$$\dot{C}_{Nk} = C_{Nk}\tilde{\omega}_k^T \tag{3}$$

where the tilde over a vector such as the angular velocity $\underset{\sim}{\omega}_k = [\omega_x \;\; \omega_y \;\; \omega_z]^T$ denotes a skew symmetric matrix of the form

$$\tilde{\omega} = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix} \tag{4}$$

The kinetic energy of body i can be expressed as

$$T_i = \frac{1}{2} \int_{m_i} \dot{\underset{\sim}{r}}_i^T \dot{\underset{\sim}{r}}_i \; dm_i \tag{5}$$

Substituting Eqs. (2) and (3), into Eq. (5), we obtain the kinetic energy of body i or

$$T_i = \frac{1}{2} m_i \left[\dot{\underset{\sim}{R}}^T \dot{\underset{\sim}{R}} + 2\sum_{j=1}^{i-1} \dot{\underset{\sim}{R}}^T C_{Nj}\tilde{\omega}_j^T \underset{\sim}{L}_j + \sum_{k=1}^{i-1} \sum_{j=1}^{i-1} \underset{\sim}{\omega}_k^T \tilde{L}_k^T C_{kj} \tilde{L}_j \underset{\sim}{\omega}_j \right]$$

$$+ \dot{\underset{\sim}{R}} C_{Ni} \tilde{S}_i \underset{\sim}{\omega}_i + \frac{1}{2} (\underset{\sim}{\omega}_i^T I_i \underset{\sim}{\omega}_i) + \sum_{j=1}^{i-1} \left[\underset{\sim}{\omega}_j^T \tilde{L}_j^T C_{ji} \tilde{S}_i \underset{\sim}{\omega}_i \right] \tag{6}$$

where the first and second moments of inertia $\underset{\sim}{S}_i$ and I_i of body i are defined as

$$\underset{\sim}{S}_i = \int_{m_i} \underset{\sim}{\rho}_i \; dm_i \qquad \underset{\sim}{\omega}_i^T I_i \underset{\sim}{\omega}_i = \int_{m_i} |\underset{\sim}{\omega}_i \times \underset{\sim}{\rho}_i|^2 \; dm_i \tag{7a,b}$$

The potential energy is due to the effects of gravity. Letting $\underset{\sim}{g}$ represent the constant gravitational acceleration vector, the gravitational potential energy of body i can be expressed as

$$V_{ig} = \int_{m_i} (\underset{\sim}{r}_i \cdot \underset{\sim}{g}) \; dm_i \tag{8}$$

Introducing Eq. (1) into Eq. (8), the gravitational potential energy of body i can be expressed in the following form:

$$V_{ig} = \int_{m_i} \left(\underset{\sim}{R}^T \underset{\sim}{g} + \sum_{j=1}^{i-1} \underset{\sim}{L}_j^T C_{jN} \underset{\sim}{g} + \underset{\sim}{\rho}_i^T C_{iN} \underset{\sim}{g} \right) dm_i \tag{9}$$

Let $\underset{\sim}{M}_i$ denote the relative joint torque at joint i applied to link i and to link i-1 in the opposite direction. It shall prove convenient to represent the virtual work in terms of various coordinates. The system's virtual work due to joint moments can be expressed in the following forms:

$$\delta W_M = \underset{\sim}{M}_1^T \delta\underset{\sim}{\theta}_1 + \underset{\sim}{M}_2^T \delta\underset{\sim}{\theta}_2 + \ldots + \underset{\sim}{M}_n^T \delta\underset{\sim}{\theta}_n \tag{10a}$$

$$= (\underset{\sim}{M}_1^T - \underset{\sim}{M}_2^T C_{12}^T)\delta\underset{\sim}{\beta}_1 + (\underset{\sim}{M}_2^T - \underset{\sim}{M}_3^T C_{23}^T)\delta\underset{\sim}{\beta}_2 + \ldots + \underset{\sim}{M}_n^T \delta\underset{\sim}{\beta}_n \tag{10b}$$

$$= (\underset{\sim}{M}_1^T - \underset{\sim}{M}_2^T C_{12}^T) D_1 \delta\underset{\sim}{\gamma}_1 + (\underset{\sim}{M}_2^T - \underset{\sim}{M}_3^T C_{23}^T) D_2 \delta\underset{\sim}{\gamma}_2 + \ldots + \underset{\sim}{M}_n^T D_n \delta\underset{\sim}{\gamma}_n \tag{10c}$$

where D_i is a matrix function of Euler angles defined in terms of inertial Euler angles measured from the inertial frame as

$$\underset{\sim}{\omega}_i = D(\underset{\sim}{\gamma}_i)\dot{\underset{\sim}{\gamma}}_i \quad \text{or} \quad \delta\underset{\sim}{\beta}_i = D(\underset{\sim}{\gamma}_i)\delta\underset{\sim}{\gamma}_i \tag{11a,b}$$

or defined in terms of relative Euler angles measured from the i-1 reference frame as

$$\underset{\sim}{\Omega}_i = D(\underset{\sim}{\alpha}_i)\dot{\underset{\sim}{\alpha}}_i \quad \text{or} \quad \delta\underset{\sim}{\theta}_i = D(\underset{\sim}{\alpha}_i)\delta\underset{\sim}{\alpha}_i \tag{12a,b}$$

where $\underset{\sim}{\omega}_i$ and $\underset{\sim}{\Omega}_i$ are the inertial and relative angular velocity vectors, $\underset{\sim}{\beta}_i$ and $\underset{\sim}{\theta}_i$ are inertial and relative quasi-coordinate vectors, and $\underset{\sim}{\gamma}_i$ and $\underset{\sim}{\alpha}_i$ are sets of inertial and relative Euler angles, respectively.

Let $\underset{\sim}{F}_i$ denote an external force vector applied at point P on link i which corresponds to the position vector $\underset{\sim}{r}_i$ of Eq. (1). Based on the variational form of Eq. (2), the virtual work due to force $\underset{\sim}{F}_i$ can be expressed as

$$\delta W_{F_i} = \underset{\sim}{F}_i^T C_{iN} \delta\underset{\sim}{R} + \underset{\sim}{F}_i^T \left[\sum_{j=1}^{i-1} C_{ij} \tilde{L}_j \, \delta\underset{\sim}{\beta}_j \right] + \underset{\sim}{F}_i^T \tilde{\rho}_i \delta\underset{\sim}{\beta}_i \tag{13}$$

Summing Eqs. (10c) and (13), the total virtual work for the system can be expressed as

$$\delta W = \underset{\sim}{F}^T \delta \underset{\sim}{R} + \sum_{i=1}^{n} \underset{\sim}{\tau}_i^T D_i \delta \underset{\sim}{\gamma}_i \tag{14}$$

where the total inertial force vector $\underset{\sim}{F}$ which causes translation of the system and the moment vector $\underset{\sim}{\tau}_i$ applied to joint i caused by joint torques and external forces can be expressed as

$$\underset{\sim}{F} = \sum_{i=1}^{n} C_{Ni} \underset{\sim}{F}_i \tag{15}$$

$$\underset{\sim}{\tau}_i = \underset{\sim}{M}_i - C_{i,i+1} \underset{\sim}{M}_{i+1} + \tilde{\rho}_i^T \underset{\sim}{F}_i + \sum_{j=i+1}^{n} \tilde{L}_i^T C_{ij} \underset{\sim}{F}_j \tag{16}$$

The total kinetic and potential energies for the multibody system are the sums of the energies for each of the n bodies or

$$T = \sum_{i=1}^{n} T_i \qquad V = \sum_{i=1}^{n} V_i \tag{17a,b}$$

Note that the kinetic energy of body i is a function of the velocity of the base and the orientations and angular velocities of the inboard links or $T_i = T_i(\dot{\underset{\sim}{R}}, C_{N1}, C_{N2}, \ldots, C_{Ni}, \underset{\sim}{\omega}_1, \underset{\sim}{\omega}_2, \ldots, \underset{\sim}{\omega}_i)$. Also, the potential energy of body i is a function of the position of the base and the orientation of the inboard links or $V_i = V_i(\underset{\sim}{R}, C_{N1}, C_{N2}, \ldots, C_{Ni})$. Based on these observations, Lagrange's equations of motion for the system can be expressed in the following quasi-coordinate form (Quinn, 1990):

$$\sum_{i=1}^{n} \frac{d}{dt}\left[\frac{\partial T_i}{\partial \dot{\underset{\sim}{R}}}\right] + \frac{\partial V_i}{\partial \underset{\sim}{R}} = \underset{\sim}{F} \tag{18a}$$

$$\frac{d}{dt}\left[\frac{\partial T_i}{\partial \underset{\sim}{\omega}_i}\right] + \tilde{\omega}_i^T \frac{\partial T_i}{\partial \underset{\sim}{\omega}_i} + \frac{\partial (V_i - T_i)}{\partial \underset{\sim}{\beta}_i} +$$

$$\sum_{j=i+1}^{n} \left\{\frac{d}{dt}\left[\frac{\partial T_j}{\partial \underset{\sim}{\omega}_i}\right] + \tilde{\omega}_i^T \frac{\partial T_j}{\partial \underset{\sim}{\omega}_i} + \frac{\partial (V_j - T_j)}{\partial \underset{\sim}{\beta}_i}\right\} = \underset{\sim}{\tau}_i \qquad i = 1, 2, \ldots, n \tag{18b}$$

where

$$\frac{\partial (\ )}{\partial \underset{\sim}{\beta}_i} = D_i^{-T} \frac{\partial (\ )}{\partial \underset{\sim}{\gamma}_i} \tag{18c}$$

Note that Eqs. (18) need only be formulated for the general body i, at which time the equations for all n bodies are known based on the general form of these equations. Also, note that the part of Eq. (18b) for body i which concerns the energies of an outboard body j need be formulated only once for a general body j. The summation satisfies the inclusion of all outboard bodies. This formulation has been applied to the planar n body problem as a derivative of this work (Silverberg, 1991).

Alternatively, we could have expressed the rotational equations of motion, Eq. (18b), in terms of true coordinates. It has been shown (Quinn, 1990), that this is equivalent to premultiplying Eq. (18b) by D_i^T or

$$\frac{d}{dt}\left[\frac{\partial T_i}{\partial \dot{\gamma}_i}\right] + \frac{\partial (V_i - T_i)}{\partial \gamma_i} + \sum_{j=i+1}^{n}\left\{\frac{d}{dt}\left[\frac{\partial T_j}{\partial \dot{\gamma}_i}\right] + \frac{\partial (V_j - T_j)}{\partial \gamma_i}\right\} = D_i^T \tau_i \qquad i = 1, 2, \ldots, n \tag{19}$$

This is clear from a comparison of the virtual work expressions of Eqs. (10b and 10c). However, this formulation results in tedious mathematics because these coordinates are not orthogonal.

The following identities are very helpful in the implementation of Eqs. (18) to formulate the equations of motion in matrix form (Quinn, 1990):

$$D^{-T}\left[a^T \frac{\partial C}{\partial \gamma} b\right] = \tilde{a}^T C b \tag{20}$$

$$\tilde{a}^T \tilde{b} + \tilde{b}\tilde{a} + \tilde{[\tilde{a}b]} = 0 \tag{21}$$

where a and b are arbitrary vectors and the second tilde over the symbol (a) represents a skew operation on the vector $[\tilde{a}b]$. Equation (20) permits a straightforward and relatively effortless formulation of the terms in Eq. (18b) which are defined by Eq. (18c). In fact, the matrix D is not developed or used except symbolically in this formulation. After simplifications based on Eq. (21), the rigid-body equations of motion can be expressed as

$$\sum_{i=1}^{n}\left\{m_i \ddot{R}_i + C_{Ni} A_i(S) + m_i \sum_{j=1}^{i-1}\left[C_{Nj} A_j(L)\right] + m_i g\right\} = F \tag{22a}$$

$$\tilde{S}_i^T C_{iN} \ddot{R} + I_i \dot{\omega}_i + \tilde{\omega}_i^T I_i \omega_i + \tilde{S}_i^T C_{iN} g + \tilde{S}_i^T \sum_{j=1}^{i-1}\left\{C_{ij} A_j(L)\right\} +$$

$$\tilde{L}_i^T \sum_{j=i+1}^{n}\left\{m_j C_{jN} \ddot{R} + m_j \sum_{k=1}^{j-1} C_{ik} A_k(L) + C_{ij} A_j(S) + m_j \tilde{L}_i^T C_{iN} g\right\} = \tau_i$$

$$i=1,\ 2,\ \ldots,\ n \tag{22b}$$

where a vector acceleration operator $A(\ )$ has been defined such that

$$A_i(b) = (\tilde{\omega}_i^2 + \dot{\tilde{\omega}}_i^T) b \tag{22c}$$

where b is a typical vector. Equation (22a) governs the translational motion of the system and Eq. (22b) governs the rotational motion of each of the n bodies. Note that these equations are highly nonlinear for high speed rotational motion because they include Coriolis and centrifugal effects.

3. Inertial Versus Relative Coordinates

We could develop equations governing the rotational motion of the n body system in terms of relative quasi-coordinates with the use of Lagrange's equations in the form of Eq. (18b) with ω replaced by Ω, β replaced with θ, τ replaced with M and

$$\frac{\partial(\ )}{\partial \theta_i} = D_i^{-T} \frac{\partial(\ )}{\partial \alpha_i} \tag{23}$$

where external forces are neglected so that the virtual work can be expressed as in Eq. (10a). This method has been implemented to formulate the "relative" equations of motion (Chang, 1987).

Alternatively, we develop equations of motion in terms of relative quasi-coordinates based on a comparison of virtual work expressions in terms of relative and inertial quasi-coordinates, Eqs. (10a) and (10b), respectively. From Eqs. (10b) and (18b), the n sets of equations governing rotational motion of the n bodies in terms of inertial quasi-coordinates take the form:

$$\underset{\sim}{g}_1(\dot{\underset{\sim}{\omega}}, \underset{\sim}{\omega}, C) = \underset{\sim}{M}_1 - C_{12}\underset{\sim}{M}_2 \tag{24a}$$

$$\underset{\sim}{g}_2(\dot{\underset{\sim}{\omega}}, \underset{\sim}{\omega}, C) = \underset{\sim}{M}_2 - C_{23}\underset{\sim}{M}_3 \tag{24b}$$

$$\vdots$$

$$\underset{\sim}{g}_{n-1}(\dot{\underset{\sim}{\omega}}, \underset{\sim}{\omega}, C) = \underset{\sim}{M}_{n-1} - C_{n-1,n}\underset{\sim}{M}_n \tag{24c}$$

$$\underset{\sim}{g}_n(\dot{\underset{\sim}{\omega}}, \underset{\sim}{\omega}, C) = \underset{\sim}{M}_n \tag{24d}$$

where $\underset{\sim}{g}_i$ represents the left side of Eq. (18b) for body i, $\underset{\sim}{\omega}$ is a vector of 3n inertial angular velocities defined as $\underset{\sim}{\omega}^T = [\underset{\sim}{\omega}_1^T \; \underset{\sim}{\omega}_2^T \; \dots \; \underset{\sim}{\omega}_n^T]$ and C represents the set of coordinate rotation matrices which describe the orientations of the bodies. Solving Eqs. (24) for $\underset{\sim}{M}_i$, the relative quasi-coordinate equations of motion for body i can be expressed in the form:

$$\underset{\sim}{M}_i = \underset{\sim}{g}_i + C_{i,i+1}\underset{\sim}{g}_{i+1} + C_{i,i+2}\underset{\sim}{g}_{i+2} + \dots + C_{i,n}\underset{\sim}{g}_n \qquad i = 1, 2, \dots, n \tag{25}$$

or in matrix form

$$\underset{\sim}{M} = G^T\underset{\sim}{g} \tag{26}$$

where the system transformation and connectivity matrix is defined as

$$G = \begin{bmatrix} C_{11} & & & & & \\ C_{21} & C_{22} & & & 0 & \\ C_{31} & C_{32} & C_{33} & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ C_{n1} & C_{n2} & \cdot & \cdot & \cdot & C_{nn} \end{bmatrix} \tag{27}$$

also, $\underset{\sim}{M}^T = [\underset{\sim}{M}_1^T \; \underset{\sim}{M}_2^T \dots \underset{\sim}{M}_n^T]$ and $\underset{\sim}{g}^T = [\underset{\sim}{g}_1^T \; \underset{\sim}{g}_2^T \dots \underset{\sim}{g}_n^T]$. Each block element C_{ij} in Eq. (27) is a 3 by 3 matrix that rotates frame j to frame i and C_{ii} is an identity matrix.

Examining Eq. (26), it is clear that the equations in terms of relative coordinates consist of a combination of the equations in terms of inertial coordinates and contain many more terms. Hence, the "inertial" equations require less effort to formulate and are more computationally efficient. Also, if one wishes to formulate the "relative" equations, a convenient method is to formulate the "inertial" equations and then assemble the "relative" equations as described by Eq. (26). Note that these "relative" equations are expressed in terms of the inertial angular velocities $\tilde{\omega}$. To complete the transformation of the equations of motion from inertial to relative quasi-coordinates, $\tilde{\omega}$ and $\dot{\tilde{\omega}}$ need to be transformed to $\tilde{\Omega}$ and $\dot{\tilde{\Omega}}$ as is done in the following section.

4. Incorporation of Joint Constraints: Simulation Method

The equations of motion in the form of Eq. (22) along with the kinematic equations given by Eq. (11a) are suitable for simulation and control of an n-body system where each joint connecting the bodies has three degrees-of-freedom. If any of the joints have less than three degrees of freedom, these equations remain valid for control purposes. Given the desired joint angular displacements, velocities and accelerations as functions of time, Eq. (22) may be solved algebraically to determine the required joint moments which would cause the desired motion. In joints with less than three degrees of freedom, these moments include constraint moments which may be discarded or used to monitor joint loading. Based on the discussion in the previous section, this may be computationally efficient despite the excess information that is generated. However, if one is concerned with dynamic simulation given joint torques, kinematic equations must be included to constrain non feasible joint motions.

The kinematic equations which describe the freedom of motion permitted by joint i may be expressed in velocity form as in Eq. (12a) or

$$\tilde{\Omega}_i = D(\tilde{\alpha}_i)\dot{\tilde{\alpha}}_i \qquad (28)$$

where the definition of $\tilde{\Omega}_i$ remains the same, the angular velocity vector of body i relative to body i-1. However, in this case, $\tilde{\alpha}_i$ is the set of d_i joint coordinates, where d_i is the number of degrees of freedom of joint i (d_i = 1, 2 or 3), and the D_i matrix is of dimension 3 by d_i. The total degrees of freedom of the system d_t is, therefore, equal to the sum of the degrees of freedom of the joints, or $d_t = d_1 + d_2 + \ldots + d_n$.

If we wish the equations of motion to be of minimum order, Eq. (22b) may be transformed into configuration space and the kinematic constraint equations may be incorporated such that constrained coordinates may be eliminated.

The equations describing the rotational motions of the links in terms of inertial quasi-coordinates, Eq. (22b), can be rearranged and expressed in the following general form:

$$A\dot{\tilde{\omega}} = \tilde{f}(\tilde{\omega},\tilde{C},\tilde{\tau}) \tag{29}$$

where A is an inertia matrix of dimension 3n by 3n and $\tilde{\tau}$ is a vector of input torques defined as $\tilde{\tau}^T = [\tilde{\tau}_1^T, \tilde{\tau}_2^T, \ldots, \tilde{\tau}_n^T]$.

The equations relating inertial versus relative accelerations and velocities may be expressed as

$$\dot{\tilde{\omega}}_i = \sum_{j=1}^{i} (\tilde{\omega}_i C_{ij} + C_{ij}\tilde{\omega}_j^T)\tilde{\Omega}_j + C_{ij}\dot{\tilde{\Omega}}_j \qquad \tilde{\omega}_i = \sum_{j=1}^{i} C_{ij}\tilde{\Omega}_j \tag{30a,b}$$

The global forms of Eqs. (30) can be expressed as

$$\dot{\tilde{\omega}} = G\dot{\tilde{\Omega}} + \tilde{h}(\tilde{\omega},\tilde{\Omega},C) \qquad \tilde{\omega} = G\tilde{\Omega} \tag{31a,b}$$

where G is the global coordinate transformation matrix defined by Eq. (27), $\tilde{\Omega}$ is a vector of all n body relative angular velocity vectors or $\tilde{\Omega}^T = [\tilde{\Omega}_1^T, \tilde{\Omega}_2^T, \ldots, \tilde{\Omega}_n^T]$ and $\tilde{h}(\tilde{\omega},\tilde{\Omega},C)$ incorporates the first term on the right side of Eq. (30a) for all of the bodies.

Introducing Eqs. (31a,b) into Eq. (29) and premultiplying by G^T (based on the discussion in Section 3), we have governing equations for rotational motion of the n bodies in terms of relative quasi-coordinates or

$$B\dot{\tilde{\Omega}} = \tilde{f}'(\tilde{\Omega},\tilde{C},\tilde{M}) \tag{32}$$

where $B = G^T A G$ is a 3n by 3n symmetric inertia matrix and $\tilde{f}'$ is a vector function of the relative angular velocities, orientations and joint moments.

Equation (28) describes the kinematic constraints at joint i where $\tilde{\alpha}_i$ is the set of joint coordinates. Taking the time derivative of that equation, we obtain

$$\dot{\tilde{\Omega}}_i = \dot{D}_i(\tilde{\alpha}_i)\dot{\tilde{\alpha}}_i + D_i(\tilde{\alpha}_i)\ddot{\tilde{\alpha}}_i \tag{33}$$

The global forms of Eqs. (33) and (28) can be expressed as

$$\dot{\tilde{\Omega}} = \dot{D}\,\dot{\tilde{\alpha}} + D\,\ddot{\tilde{\alpha}} \qquad \tilde{\Omega} = D\,\dot{\tilde{\alpha}} \tag{34a,b}$$

where $\tilde{\alpha}$, the set of all joint coordinates, contains d_t elements and the 3n by d_t matrix $D = D(\tilde{\alpha})$ is defined as a block diagonal global transformation matrix from the joint coordinates to the relative quasi-coordinates or

$$D = \begin{bmatrix} D_1 & & & 0 \\ & D_2 & & \\ & & \ddots & \\ 0 & & & D_n \end{bmatrix} \tag{35}$$

Introducing Eqs. (34a,b) into Eq. (32), the equations of motion can be expressed as

$$BD\ddot{\tilde{\alpha}} = \tilde{f}''(\dot{\tilde{\alpha}},\tilde{C},\tilde{M}) \tag{36}$$

where the vector function $\tilde{f}''$ includes the term $\dot{D}\dot{\tilde{\alpha}}$ from Eq. (34a) and is a function of joint coordinate velocities, body orientations and joint torques.

Recall that in Section 2 Lagrange's Equations could be premultiplied by a matrix D^{-T} to switch from true coordinates to quasi-coordinates. We now use this concept to complete the transformation of the equations of motion back to true coordinates. Premultiplying Eq. (36) by D^T, the equations describing relative rotational motion of the n body system in terms of the true coordinates can be expressed as

$$M_c\ddot{\underset{\sim}{\alpha}} = D^T\underset{\sim}{f}''(\dot{\underset{\sim}{\alpha}},C,\underset{\sim}{M}) \tag{37}$$

The matrix $M_c = D^TBD$ is the square d_t dimensional inertia matrix in configuration space which is a function of orientations of the n bodies. Inverting the inertia matrix and premultiplying Eq. (37) by the result, the acceleration of the joint coordinates can be expressed as

$$\ddot{\underset{\sim}{\alpha}} = M_c^{-1}D^T\underset{\sim}{f}''(\dot{\underset{\sim}{\alpha}},C,\underset{\sim}{M}) \tag{38}$$

Equation (38) represents the equations of motion of the multibody system in terms of the true joint coordinates. Hence, these equations are of minimum order. Alternatively, we could have arrived at these equations by means of Lagrange's equations in terms of the joint coordinates. However, applying Lagrange's equations in true coordinates to model the three dimensional rotational motion of bodies is well known to require a great deal of effort. In contrast, we have shown that forming Lagrange's Equations in terms of inertial quasi-coordinates is comparatively straightforward. The additional effort to transform the equations into configuration space is also relatively minimal and the process is straightforward.

Given the joint angular displacements and velocities and the control moments at a time step, numerical integration of Eq. (38) in state space yields the joint displacements and velocities at the next time step. Note that this method of solving the equations of motion is valid for joints with one, two or three degrees of freedom and, hence, for n body systems with total number of degrees of freedom between n and 3n or, $n \le d_t \le 3n$.

In implementation it is not necessary or desirable to transform all of the variables explicitly from inertial quasi-coordinates to joint coordinates. The joint accelerations can be expressed in the mixed variable form:

$$\ddot{\underset{\sim}{\alpha}} = M_c^{-1}\left[D^TG^T\underset{\sim}{f}(\underset{\sim}{\omega},C,\underset{\sim}{\tau}) - D^TG^TA\ \underset{\sim}{h}(\underset{\sim}{\omega},\underset{\sim}{\Omega},C) - D^TG^TAG\dot{D}\dot{\underset{\sim}{\alpha}}\right] \tag{39}$$

The equations of motion in the form of Eq. (39) along with Eqs. (31b) and (34b) were implemented for the purpose of performing numerical examples.

5. Numerical Examples

First, the control problem is solved: the required joint torques are determined which will cause the desired joint motions. To this end, we solve Eq. (22), the equations of motion in terms of inertial quasi-coordinates; that is, we algebraically compute $\tilde{\tau}_i$, i=1,2, ...,n, and then compute the joint torques $\tilde{M} = G^T\tilde{\tau}$.

In the simulation problem, the joint torques and the initial values of the joint state are inputs for the equations of motion, Eq. (39). The time histories of each joint's displacement and velocity are to be determined for a prescribed input torque history.

Example One

There are two examples presented in this paper, the first is the simplest multi-body problem: a two-link, two-joint, two DOF planar manipulator shown in Fig. 2. Each joint has a single DOF in the z direction. The local coordinate axes are defined so that the x-axis coincides with the link axis of symmetry, the z-axis is parallel to the Z-axis. The two uniform, slender links have the same mechanical properties: each link's mass and length are 1.754kg and 0.915 meters.

The desired trajectory of the end-effector is a straight line from point (0.305,0.305,0) meters to point (0.915,0.915,0) meters. The end-effector acceleration along this line is specified as cos(t) for t=0 to π seconds. A solution of the inverse kinematics problem determined the joints' angular accelerations, velocities and displacements.

The required joint torques were determined as discussed above. Using these joint torques, a simulation was performed with two integration time step