

EQUATIONS OF MOTION FOR STRUCTURES
IN TERMS OF QUASICOORDINATES

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ABSTRACT

A form of Lagrange's Equations in terms of quasicordinates (Boltzmann/Hamel equations) is presented. Identities are introduced which permit a straightforward formulation of the equations of motion for structures for which the kinetic and potential energies are functions of angular velocity and orientation. The formalism is presented in matrix form and may be used if the energies are expressed in matrix form as explicit functions of angular velocities and coordinate transformation matrices. This method is particularly useful for a large class of problems in the dynamics of structures including spacecraft, robots, ground vehicles and aircraft.

1. Introduction

The purpose of this paper is to present a form of Lagrange's equations in terms of quasicordinates (Boltzmann/Hamel equations). This form is specific to a large class of problems in the dynamics of structures including spacecraft, robots, ground vehicles and aircraft where the angular velocities of the structure can be considered to be time derivatives of quasicordinates. Identities are introduced which permit a straightforward method of formulating the equations of motion when the kinetic and potential energies are explicit functions of angular orientation. The formulation is presented in matrix form so that the kinetic and potential energies need to be expressed in matrix form as functions of angular velocities and coordinate transformation matrices.

The concept of quasi-coordinates is not new. According to Whittaker (1944) and Neimark and Fufaev (1967), Lagrange and Euler used quasicordinates to study rigid body motion and the so-called Lagrange's equations for quasicordinates were developed by Boltzmann and Hamel at the beginning of the twentieth century. Hence, this Lagrangian formalism is sometimes called the Boltzmann/Hamel equations. Advanced texts in dynamics such as those by Whittaker (1944) and Meirovitch (1970) include a section on the subject; and quasicordinate formulations have been recently used by Passeron et. al. (1986), Huston and Passerello (1980), and Oz et. al. (1980). However, the treatment in dynamics texts is usually of a general nature and, as such, a simplification which enhances the utility of the method does not appear in the literature.

Lagrange's equations are widely used because they provide (i) a

straightforward and orderly analytical approach based on a single scalar function which produces (ii) form-invariant equations of motion in which (iii) holonomic constraint forces can be eliminated, included or retrieved, as desired, and (iv) natural symmetry is preserved. According to Neimark and Fufaev (1967), Lagrange' equations for quasicordinates were developed as a form which was uniformly valid for all dynamic systems with or without constraints for true or quasi-coordinates. In practical terms, Lagrange's equations for quasicordinates permit a most efficient formulation of the equations of motion of some systems. Perhaps, the clearest (and earliest) example is that of a structure undergoing finite rotations in three-dimensional space. If true generalized coordinates are chosen to represent these rotations, the derivation of the equations of motion using Lagrange's equations becomes quite tedious. In the quasicordinate formulation, the angular velocity vector components are the time derivatives of a set of quasicordinates rather than true coordinates; these quasicordinates are defined only in terms of their differentials. This approach is based on the observation that the kinetic energy of a body can be represented in its most compact form in terms of these angular velocity components or quasivelocities.

It is important to note that the validity of the equations of motion of a dynamic system depends on the system model used in their development, not on the particular method of formulation, as there are many theoretically equivalent methods (all based on Lagrange's principle of virtual work) as discussed by Likins (1974, 1975). However, the form of the resulting equations may be different and considerable mathematical manipulations may be necessary for comparisons. The best

method of equation of motion development for a particular dynamic system produces the simplest and most useful (e.g. symmetry preserved) form of the equations in an orderly fashion. It is for this reason of formulation efficiency that the Boltzmann/Hamel equation has been examined.

General forms of Lagrange's equations in quasicoordinate form appear ungainly. It is when special cases are examined that these equations appear most promising. An important special case is that mentioned earlier where a structure undergoes finite rotations and the kinetic and potential energies are not functions of the angular orientation. The purpose of this paper is to present simplifying identities which enhance the usefulness of this approach in cases where the energies are explicit functions of the angular orientation.

In the kinematic analysis of a structure, it is often convenient to express different velocity and position vectors in different reference frames rather than all in the same frame. Hence, when the energies are expressed in matrix form, coordinate transformation matrices must be introduced. These coordinate transformation matrices and, hence, the energies are functions of the relative angular orientation of the reference frames. For this reason, a quasicoordinate form of Lagrange's equations which can easily account for terms involving angular orientation is valuable.

This formulation of Lagrange's equations for quasicoordinates is clearly useful for the dynamic analysis of robots, spacecraft, ground vehicles and aircraft. The equations governing the rotational motion of a maneuvering flexible spacecraft are developed using this approach. The purpose of this example is to demonstrate this method of dynamic

analysis for a problem of current interest, the solution of which has been published previously, for comparison.

2. General Form of Lagrange's Equations For Quasicoordinates

Lagrange's equations of motion for holonomic systems in terms of true generalized coordinates can be expressed in the matrix form

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = \underline{Q} \quad (1)$$

where q is a set of n true independent generalized coordinates, T and V are the kinetic and potential energies where $T=T(q, \dot{q})$, $V=V(q)$ and \underline{Q} is a set of generalized forces which are defined in terms of the virtual work expression

$$\delta W = \underline{Q}^T \delta \underline{q} \quad (2)$$

Lagrange's equations in the form of Eq. (1) provide a straightforward method of deriving reactionless equations of motion for dynamic systems. However, when a structure is free to undergo finite rotations, derivation of the equations governing the rigid-body orientation in this manner can become tedious. For this reason other forms of Lagrange's equations have been developed.

The most compact form of the kinetic energy of a structure undergoing finite rotations is expressed in terms of the inertial angular velocity vector. Hence, we might conclude that the most efficient derivation of the equations of motion must also be in terms of these angular velocities. For this reason a concept was developed where the angular velocity vector is considered to be the time derivatives of so-called quasicoordinates which are of themselves undefined.

That part of Eq. (1) which governs the orientation of a structure

can be expressed as

$$\frac{d}{dt} \left(\frac{\partial \bar{T}}{\partial \dot{\underline{\alpha}}} \right) - \frac{\partial \bar{T}}{\partial \underline{\alpha}} + \frac{\partial V}{\partial \underline{\alpha}} = \underline{Q}_{\alpha} \quad (3)$$

where $\underline{\alpha}$ is a set of three true coordinates such as Euler angles and \underline{Q}_{α} is a set of corresponding moments. The angular velocity vector can be expressed as a function of the true coordinates and their time derivatives or in matrix form

$$\underline{\omega} = D(\underline{\alpha}) \dot{\underline{\alpha}} \quad (4)$$

where $D(\underline{\alpha})$ is a 3x3 matrix which is a nonlinear function of $\underline{\alpha}$. The kinetic energy can then be expressed as an explicit function of $\underline{\omega}$ and $\underline{\alpha}$, or $\bar{T}(\underline{\omega}, \underline{\alpha})$. The introduction of the inverse of Eq. (4) into Eq. (3) permits Lagrange's equations to be represented in terms of $\underline{\omega}$. In this manner Lagrange's equations of motion for rigid body rotations can be expressed as

$$D^{-1} \frac{d}{dt} \left(\frac{\partial \bar{T}}{\partial \underline{\omega}} \right) + \dot{D}^{-1} \left(\frac{\partial \bar{T}}{\partial \underline{\omega}} \right) - \underline{\omega}^T D^{-1} \frac{\partial D}{\partial \underline{\alpha}} \frac{\partial \bar{T}}{\partial \underline{\omega}} - \frac{\partial \bar{T}}{\partial \underline{\alpha}} + \frac{\partial V}{\partial \underline{\alpha}} = \underline{Q}_{\alpha} \quad (5)$$

where the following matrix notation has been used:

$$\frac{\partial D}{\partial \underline{\alpha}} D^{-1} \underline{\omega} = \left[\begin{array}{ccc} \left[\frac{\partial D}{\partial \alpha_1} \right] D^{-1} \underline{\omega} & \left[\frac{\partial D}{\partial \alpha_2} \right] D^{-1} \underline{\omega} & \left[\frac{\partial D}{\partial \alpha_3} \right] D^{-1} \underline{\omega} \end{array} \right] \quad (6)$$

and $D^{-T} = (D^T)^{-1}$.

Premultiplying Eq. (5) by D^{-T} , Lagrange's equations for quasicordinates can be expressed as

$$\frac{d}{dt} \left(\frac{\partial \bar{T}}{\partial \underline{\omega}} \right) + D^{-T} \left[\dot{D}^{-1} - \underline{\omega}^T D^{-1} \frac{\partial D}{\partial \underline{\alpha}} \right] \frac{\partial \bar{T}}{\partial \underline{\omega}} + D^{-T} \left(\frac{\partial V}{\partial \underline{\alpha}} - \frac{\partial \bar{T}}{\partial \underline{\alpha}} \right) = D^{-T} \underline{Q}_{\alpha} \quad (7)$$

This form of Lagrange's equations appears in Whittaker (1944) (in indicial notation), Meirovitch (1970) and Likins (1975). Although, in the development of Eq. (7), rotating structures have been discussed, the resulting equations are of general utility in that the symbols $\underline{\alpha}$ and $\underline{\omega}$

could take other meanings. Equation (7) is actually more complex than Eq. (3) and, in practice, can be rather cumbersome to apply. Fortunately, for the class of problems considered in this paper Eq. (7) can be simplified greatly.

3. Quasicoordinate Formulations for Rotating Structures

In the case of structures undergoing finite rotations in three dimensional space, the angular velocities can be further related to the matrix D to simplify Eq. (7).

Let the matrix $C(\underline{\alpha})$ denote the orthogonal rotational transformation of a structure from the inertial reference frame to a body-fixed reference frame or $\underline{v}_B = C \underline{v}_N$ where \underline{v} is an arbitrary vector and the subscripts B and N represent the body-fixed and inertial frames, respectively. The time rate of change of this matrix can be expressed as

$$\dot{C} = \tilde{\omega} C \quad (8a)$$

where $\tilde{\omega}$ is a skew symmetric matrix defined as

$$\tilde{\omega} = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix} \quad (8b)$$

Solving Eq. (8a) for $\tilde{\omega}$, we have

$$\tilde{\omega} = \dot{C}C^T = \left[\frac{\partial C}{\partial \underline{\alpha}} \dot{\underline{\alpha}} \right] C^T \quad (9a)$$

where

$$\left[\frac{\partial C}{\partial \underline{\alpha}} \dot{\underline{\alpha}} \right] = \left[\sum_{i=1}^3 \frac{\partial C}{\partial \alpha_i} \dot{\alpha}_i \right] \quad (9b)$$

The introduction of Eq. (4) into Eq. (9a) produces an explicit relationship between the matrices C and D. This permits the proof of the following identity which is implicit in Meirovitch (1970):

$$\tilde{\omega}^T = D^{-T} \left[\dot{D}^T - \tilde{\omega}^T D^{-T} \frac{\partial D^T}{\partial \alpha} \right] \quad (10)$$

Introducing Eq. (10) into Eq. (7), Lagrange's equations for quasi-coordinates take the form

$$\frac{d}{dt} \left(\frac{\partial \bar{T}}{\partial \tilde{\omega}} \right) + \tilde{\omega}^T \frac{\partial \bar{T}}{\partial \tilde{\omega}} + D^{-T} \left(\frac{\partial V}{\partial \alpha} - \frac{\partial \bar{T}}{\partial \alpha} \right) = \underline{M} \quad (11)$$

where

$$\underline{M} = D^{-T} \underline{Q}_\alpha \quad (12a)$$

is the moment which can be defined in terms of virtual work as

$$\delta W = \underline{M}^T \delta \beta \quad (12b)$$

where β is the set of quasicordinates which are defined in differential form as $\underline{\omega} = d\beta/dt$. In the literature, attention is focused on cases where \bar{T} and V are not functions of the orientation α so that the last term on the left side of Eq. (11) is null. In this case, the result is the familiar set of equations that are often derived from angular momentum principles. On the other hand, if the last term is not null, the utilization of Eq. (11) remains rather cumbersome.

In general, the potential and kinetic energies are functions of the angular orientation of the structure. In matrix form, \bar{T} and V contain rotational transformation matrices which depend explicitly on angular orientation. A term of the kinetic energy might take the matrix form

$$\bar{T}^*(\underline{\omega}, C) = \underline{b}^T C \underline{n} \quad (13)$$

where \underline{b} and \underline{n} are vectors represented in the body-fixed and inertial reference frames, respectively. In this case, the last term on the left side of Eq. (11) involving $\bar{T}^*(\underline{\omega}, \alpha)$ can be expressed as

$$D^{-T} \frac{\partial \bar{T}^*}{\partial \alpha} = D^{-T} \left\{ \underline{b}^T \frac{\partial C}{\partial \alpha} \underline{n} \right\} \quad (14)$$

where the notation of Eq. (6) has been used to define the terms on the

right hand side of Eq. (14). Moreover, it can be shown that

$$D^{-T} \left\{ \underline{b}^T \frac{\partial C}{\partial \underline{\alpha}} \underline{n} \right\} = \underline{b}^T C \underline{n} \quad (15)$$

To show this, Eq. (15) is multiplied by $\dot{\underline{\alpha}}^T D^T$ and Eq. (4) is introduced into the result producing the expression

$$\dot{\underline{\alpha}}^T \left\{ \underline{b}^T \frac{\partial C}{\partial \underline{\alpha}} \underline{n} \right\} = \underline{b}^T \tilde{\omega} C \underline{n} \quad (16)$$

Introducing Eq. (9a) into Eq. (16) results in the expression

$$\dot{\underline{\alpha}}^T \left\{ \underline{b}^T \frac{\partial C}{\partial \underline{\alpha}} \underline{n} \right\} = \underline{b}^T \left[\frac{\partial C}{\partial \underline{\alpha}} \dot{\underline{\alpha}} \right] \underline{n} \quad (17)$$

which, considering Eq. (9b), is an identity. Equation (14) can then be expressed as

$$D^{-T} \frac{\partial \bar{T}^*}{\partial \underline{\alpha}} = \underline{b}^T C \underline{n} \quad (18)$$

or considering Eq. (13) we can introduce the following notation

$$\frac{\partial \bar{T}^*}{\partial C} = \underline{b}^T C \underline{n} \quad (19)$$

If the kinetic and potential energies are expressed in matrix form in terms of angular velocity vectors and rotational transformation matrices or $\bar{T}(\underline{\omega}, C)$ and $\bar{V}(C)$, then Lagrange's Equations for quasicordinates can be expressed as

$$\frac{d}{dt} \left(\frac{\partial \bar{T}}{\partial \underline{\omega}} \right) + \tilde{\omega}^T \frac{\partial \bar{T}}{\partial \underline{\omega}} - \frac{\partial \bar{T}}{\partial C} + \frac{\partial \bar{V}}{\partial C} = \underline{M} \quad (20)$$

The operations implicit in the last two terms on the left side of Eq. (20) are actually straightforward according to the identity given by Eqs. (13) and (19). Hence, Eq. (20) permits a straightforward and efficient method of formulating the equations of motion of complex structures.

4. Example

As an example of the application of Lagrange's equations for quasicoordinates, consider a flexible spacecraft orbiting the earth. The spacecraft of Fig. 1 consists of the shuttle, which will be assumed to be relatively rigid, and a flexible appendage extending from the shuttle cargo bay. Considering Fig. 1 and denoting the position of the origin O of the body-fixed reference frame $x_0y_0z_0$ by the vector \underline{R} , the position of a point S on the shuttle relative to O by \underline{r} , and the position of a point A on the appendage relative to O by \underline{a} . Moreover, the elastic displacement vector of point A is defined as \underline{u} . The position of S and A relative to the inertial frame XYZ is $\underline{R}_S = \underline{R} + \underline{r}$ and $\underline{R}_A = \underline{R} + \underline{a} + \underline{u}$, respectively. The velocities of points S and A on the spacecraft are $\dot{\underline{R}}_S = \dot{\underline{R}} + \underline{\omega} \times \underline{r}$ and $\dot{\underline{R}}_A = \dot{\underline{R}} + \underline{\omega} \times (\underline{a} + \underline{u}) + \dot{\underline{u}}$, respectively, where $\dot{\underline{R}}$ is the translational velocity and $\underline{\omega}$ is the angular velocity of the frame $x_0y_0z_0$ with respect to the inertial frame. Hence, the kinetic energy of the spacecraft is

$$T = \frac{1}{2} \int_{m_S} |\dot{\underline{R}}_S|^2 dm_S + \frac{1}{2} \int_{m_A} |\dot{\underline{R}}_A|^2 dm_A \quad (21)$$

In order to discretize the system in space, we express the elastic displacement vector in the form

$$\underline{u}(x, t) = \Phi(x) \underline{q}(t) \quad (22)$$

where Φ is a matrix of space-dependent admissible functions and \underline{q} is a vector of time-dependent generalized coordinates. Introducing Eq. (22) into Eq. (21), the kinetic energy can be expressed in the matrix form

$$T = \frac{1}{2} \dot{\underline{R}}^T \underline{m} \dot{\underline{R}} + \frac{1}{2} \underline{\omega}^T \underline{I}_0 \underline{\omega} + \dot{\underline{R}}^T \underline{C}^T \underline{S}_0 \underline{\omega} + \frac{1}{2} \dot{\underline{q}}^T \underline{M}_A \dot{\underline{q}} + \dot{\underline{R}}^T \underline{C}^T \underline{\Phi} \dot{\underline{q}} + \dot{\underline{R}}^T \underline{C}^T \underline{\omega}^T \underline{\Phi} \underline{q}$$

$$+ \dot{\underline{q}}^T \bar{\Phi}^T \underline{\omega} + \underline{\omega}^T \int_{m_A} \bar{a}^T \bar{\omega}^T \Phi dm_A \underline{q} - \frac{1}{2} \underline{q}^T \bar{L}_A(\underline{\omega}) \underline{q} + \dot{\underline{q}}^T \bar{L}_A(\underline{\omega}) \underline{q} \quad (23)$$

where

$$\bar{\Phi} = \int_{m_A} \Phi dm_A \quad (24a)$$

$$\bar{\Phi}^T = \int_{m_A} \Phi^T \bar{a} dm_A \quad (24b)$$

$$\bar{L}_A(\underline{\omega}) = \int_{m_A} \Phi^T \bar{\omega}^T \Phi dm_A \quad (24c)$$

$$\bar{L}_A(\underline{\omega}) = \int_{m_A} \Phi^T \bar{\omega}^2 \Phi dm_A \quad (24d)$$

$$M_A = \int_{m_A} \Phi^T \Phi dm_A \quad (24e)$$

$$\underline{S}_0 = \int_{m_S} \underline{r} dm_S + \int_{m_A} \underline{a} dm_A \quad (24f)$$

Also, m is the mass and I_0 is the mass moment of inertia matrix about point O of the spacecraft. The matrix M_A is the mass matrix of the appendage. The matrix C represents a rotational transformation from the inertial frame to the body-fixed frame.

Assuming the origin of the inertial coordinate system coincides with the center of the gravitational field, the gravitational potential energy can be expressed as

$$V_g = -Gm_e \left(\int_{m_S} |\underline{R} + \underline{r}|^{-1} dm_S + \int_{m_A} |\underline{R} + \underline{a} + \underline{u}|^{-1} dm_A \right) \quad (25)$$

where m_e is the mass of the earth and G is the gravitational constant. The strain energy can be expressed as an energy inner product denoted by $[,]$ as in Meirovitch (1980). The total potential energy then becomes

$$V = \frac{1}{2} [u, u] + V_g \quad (26)$$

where $[u, u]$ includes the potential energy due to centrifugal and gravitational stiffening effects.

Recognizing that the magnitude of \underline{R} is large in comparison with the magnitude of the other vectors in Eq. (25) and ignoring higher order terms, a binomial expansion permits us to write

$$\begin{aligned}
 V_g \cong & -Gm_o \left\{ m |\underline{R}|^{-1} - \underline{R}^T \left(\underline{S}_o + \int_{m_A} \underline{u} \, dm_A \right) |\underline{R}|^{-3} \right. \\
 & + \frac{1}{2} |\underline{R}|^{-3} \left[\int_{m_s} (3\underline{r}^T \underline{C} \underline{R} \underline{R}^T \underline{C}^T \underline{r} - \underline{r}^T \underline{r}) \, dm_s \right. \\
 & \left. \left. + \int_{m_A} (3\underline{a}^T \underline{C} \underline{R} \underline{R}^T \underline{C}^T \underline{a} - \underline{a}^T \underline{a}) \, dm_A \right] \right\} \quad (27)
 \end{aligned}$$

Introducing Eq. (27) into Eq. (26) and considering Eq. (22), the potential energy can be written in the matrix form

$$\begin{aligned}
 V \cong & \frac{1}{2} \underline{q}^T \underline{K}_A \underline{q} - Gm_o \left\{ \frac{m}{|\underline{R}|} - \frac{1}{|\underline{R}|^2} \hat{\underline{R}}^T \underline{C}^T (\underline{S}_o + \underline{\phi} \underline{q}) \right. \\
 & \left. + \frac{1}{2|\underline{R}|^3} \left[3\hat{\underline{R}}^T \underline{C}^T \underline{J} \underline{C} \hat{\underline{R}} - \left(\int_{m_s} \underline{r} \underline{r}^T \, dm_s + \int_{m_A} \underline{a} \underline{a}^T \, dm_A \right) \right] \right\} \quad (28)
 \end{aligned}$$

where

$$\underline{K}_A = [\underline{\phi}, \underline{\phi}] \quad (29a)$$

$$\hat{\underline{R}} = \frac{\underline{R}}{|\underline{R}|} \quad (29b)$$

$$\underline{J} = \int_{m_s} \underline{r} \underline{r}^T \, dm_s + \int_{m_A} \underline{a} \underline{a}^T \, dm_A \quad (29c)$$

are the stiffness matrix of the appendage, a unit vector in the direction of \underline{R} and a matrix of inertia integrals, respectively. Because the control forces are most conveniently expressed in the body-fixed frame, the transformation matrix \underline{C} must be employed in expressing the virtual work as follows:

$$\delta W = \underline{F}^T \underline{C} \delta \underline{R} + \underline{M}^T \delta \underline{\beta} + \underline{Q}^T \delta \underline{q} \quad (30)$$

where \underline{F} , \underline{M} and \underline{Q} are generalized force vectors in terms of components

about x_0 , y_0 and z_0 .

Lagrange's equations which govern the translation and elastic vibration of the spacecraft can be written in the symbolic form

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\underline{R}}} \right) + \frac{\partial V}{\partial \underline{R}} = \underline{C}^T \underline{F} \quad (31a)$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\underline{q}}} \right) - \frac{\partial T}{\partial \underline{q}} + \frac{\partial V}{\partial \underline{q}} = \underline{Q} \quad (31b)$$

whereas Eq. (20) governs the rotational motion of the structure. The equations of motion are expressed in detail by Meirovitch and Quinn (1987). The purpose of this example is to demonstrate the use of Eqs. (20) and (19) in producing the equations which govern the rotational motion of the structure.

For convenience we shall consider the ten terms of Eq. (23) describing the kinetic energy of the structure separately, or

$$T = \sum_{i=1}^{10} T_i \quad (32)$$

where the individual T_i terms are defined by the order of Eq. (23). Considering Eq. (20), the terms of Lagrange's equations involving derivatives with respect to angular velocity can be expressed as follows:

$$N_1(T_2) = I_0 \dot{\underline{\omega}} + \underline{\tilde{\omega}}^T I_0 \underline{\omega} \quad (33a)$$

$$N_1(T_3) = \underline{\tilde{S}}_0^T \underline{C} \dot{\underline{R}} + \underline{\tilde{S}}_0^T \underline{\tilde{\omega}} \underline{C} \dot{\underline{R}} + \underline{\tilde{\omega}}^T \underline{S}_0^T \underline{C} \dot{\underline{R}} \quad (33b)$$

$$N_1(T_6) = \underline{\tilde{\omega}}^T \underline{\tilde{b}}^T \underline{C} \dot{\underline{R}} + \underline{\tilde{b}}^T \underline{C} \dot{\underline{R}} + \underline{\tilde{b}}^T \underline{\tilde{\omega}} \underline{C} \dot{\underline{R}} + \underline{\tilde{b}}^T \underline{C} \dot{\underline{R}} \quad (33c)$$

$$N_1(T_7) = \underline{\tilde{\phi}} \dot{\underline{q}} + \underline{\tilde{\omega}}^T \underline{\tilde{\phi}} \dot{\underline{q}} \quad (33d)$$

$$N_1(T_8) = \underline{J} \dot{\underline{q}} + \underline{J} \dot{\underline{q}} + \underline{\tilde{\omega}}^T \underline{J} \dot{\underline{q}} \quad (33e)$$

where

$$N_1(T_1) = \frac{d}{dt} \left(\frac{\partial T_1}{\partial \underline{\omega}} \right) + \underline{\tilde{\omega}}^T \frac{\partial T_1}{\partial \underline{\omega}} \quad (34a)$$

$$\underline{b} = \underline{\tilde{\phi}} \underline{q} \quad (34b)$$

$$J = J(\underline{\omega}) = \int_{\underline{m}_A} (\underline{\tilde{a}} \underline{\tilde{\omega}} + [\underline{\tilde{a}} \underline{\tilde{\omega}}]) \underline{\phi} \, d\underline{m}_A \quad (34c)$$

Next, considering Eqs. (13) and (19), the terms of the kinetic and potential energies, as expressed by Eqs. (23), (28) and (32), involving derivatives with respect to angular orientation can be expressed as follows:

$$N_2(T_3) = [\underline{\tilde{S}} \underline{\tilde{\omega}}]^T \underline{C} \dot{\underline{R}} \quad (35a)$$

$$N_2(T_5) = \underline{\tilde{b}}^T \underline{C} \dot{\underline{R}} \quad (35b)$$

$$N_2(T_6) = [\underline{\tilde{b}} \underline{\tilde{\omega}}]^T \underline{C} \dot{\underline{R}} \quad (35c)$$

$$N_2(V) = \frac{Gm_e}{|\underline{R}|^3} [\underline{\tilde{C}} \underline{R}] \left(\underline{S}_0 - \frac{3\underline{J} \underline{C} \underline{R}}{|\underline{R}|} + \underline{b} \right) \quad (35d)$$

where the differential operator is defined as

$$N_2(T_1) = D^{-T} \frac{\partial T_1}{\partial \underline{\alpha}} = \frac{\partial T_1}{\partial \underline{C}} \quad (36)$$

The equations governing the rotational motion of the spacecraft can be found by summing Eqs. (33) and (35d) and subtracting Eqs. (35a-c). These equations can be simplified with the introduction of the following identity involving skew symmetric matrices:

$$\underline{\tilde{a}} \underline{b} + \underline{\tilde{b}}^T \underline{a} + [\underline{\tilde{b}} \underline{a}] = [0] \quad (37)$$

where \underline{a} and \underline{b} are arbitrary vectors and the second tilde over the symbol (b) denotes a skew operation on the vector $[\underline{\tilde{b}} \underline{a}]$. Lagrange's equations governing the rotational motion of the spacecraft can then be expressed

as

$$\begin{aligned}
 I_0 \dot{\underline{\omega}} + \tilde{\omega}^T I_0 \underline{\omega} + \tilde{S}_0^T \underline{C} \ddot{\underline{R}} + \tilde{\Phi} \ddot{\underline{q}} + \left[[\tilde{C} \underline{R}] \underline{\Phi} + J(\underline{\omega}) + \tilde{\omega}^T J(\underline{\omega}) + \frac{G_m}{|\underline{R}|^3} [\tilde{C} \underline{R}] \underline{\Phi} \right] \underline{q} \\
 + [\tilde{\omega}^T \underline{\Phi} + J(\underline{\omega})] \dot{\underline{q}} + \frac{G_m}{|\underline{R}|^3} [\tilde{C} \underline{R}] \left(\underline{S}_0 + 3 \bar{J} C \frac{\hat{R}}{|\underline{R}|} \right) = \underline{M} \quad (38)
 \end{aligned}$$

Equation (37) is also helpful in simplifying the equations of motion governing translation and elastic vibration which have been presented in simplified form by Meirovitch and Quinn (1987).

5. Conclusions

A form of Lagrange's Equations for quasicordinates (Boltzmann/Hamel Equations) has been presented which provides a straightforward method of formulating the equations of motion of structures when the energy expressions are explicit functions of angular orientation. An identity (Eq. 19) has been introduced which may be utilized if the energies are expressed in matrix form as functions of angular velocities and coordinate transformation matrices. This method applies to a large class of problems in the dynamics of structures including spacecraft, robotics, ground vehicles and aircraft. The formulation of the equations of motion of a maneuvering flexible spacecraft was shown to be relatively straightforward using this method. A second simplifying identity (Eq. 37) was introduced which permits the recognition and cancellation of some like terms which appear in the Lagrangian formulation. The formulation and method, including the simplifying identity are suitable for symbolic computation. This permits the dynamic analysis of complex systems.

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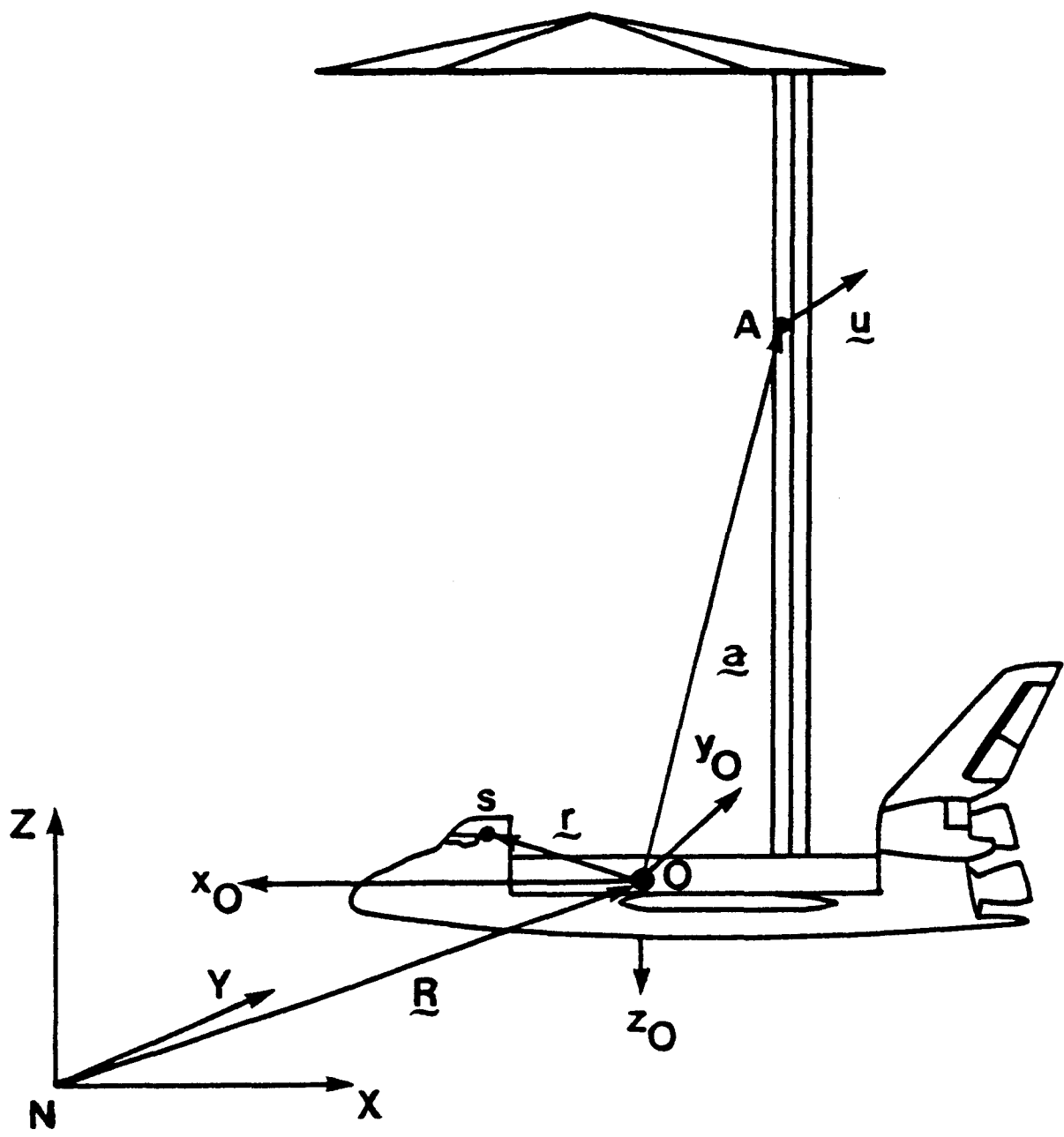


Fig. 1